

FØLNER FUNCTIONS AND THE GENERIC WORD PROBLEM FOR FINITELY GENERATED AMENABLE GROUPS

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ABSTRACT. We introduce and investigate different definitions of effective amenability, in terms of computability of Følner sets, Reiter functions, and Følner functions. As a consequence, we prove that recursively presented amenable groups have subrecursive Følner function, answering a question of Gromov; for the same class of groups we prove that solvability of the Equality Problem on a generic set (generic EP) is equivalent to solvability of the Word Problem on the whole group (WP), thus providing the first examples of finitely presented groups with unsolvable generic EP. In particular, we prove that for finitely presented groups, solvability of generic WP doesn't imply solvability of generic EP.

1. INTRODUCTION

In this paper we define and study some effective versions of amenability for finitely generated groups, in terms of computability of Følner sets, computability of Reiter functions and subrecursivity of Følner functions.

Let Γ be a group generated by a finite subset X . Recall from [4] that for any $n \in \mathbb{N}$, an n -Følner set of Γ (with respect to X) is a non-empty finite subset $\Omega \subset \Gamma$ such that

$$(1) \quad \frac{|\Omega \setminus x\Omega|}{|\Omega|} \leq n^{-1}, \quad \forall x \in X.$$

We denote by $\mathfrak{Føl}_{\Gamma,X}(n)$ the set of all n -Følner sets of Γ with respect to X . Moreover, we say that a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of subsets of Γ is a *Følner sequence* if for every $n \in \mathbb{N}$, $\Omega_n \in \mathfrak{Føl}_{\Gamma,X}(n)$. A related important notion is the Følner function $F_{\Gamma,X}$, introduced by Vershik [34], that measures the cardinality of the smallest Følner sets:

$$F_{\Gamma,X}(n) := \min\{|\Omega| : \Omega \in \mathfrak{Føl}_{\Gamma,X}(n)\},$$

with the convention that $\min \emptyset := \infty$. It is well known that the existence of a Følner sequence and the asymptotic behaviour of the function $F_{\Gamma,X}$ does not depend on the choice of X : we say that Γ is *amenable* if it admits a Følner sequence (and therefore $F_{\Gamma,X}(n) < \infty$, $\forall n \in \mathbb{N}$).

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is said to be *recursive* if there exists an algorithm (Turing machine) that:

- (i) stops for every input n ;
- (ii) *computes* f , that is, gives $f(n)$ as an output.

A function is *subrecursive* if it admits a recursive upper bound. A *partially recursive, k -place function* is a function \mathcal{U} from a domain contained in \mathbb{N}^k such that there exists an algorithm that computes \mathcal{U} and stops when the input is a string of the domain. We refer to [24] for general computability theory.

Vershik himself was interested in algorithmic behaviour of Følner functions, conjecturing the existence of arbitrarily fast growing Følner functions. This was confirmed by Erschler [10], who provided examples of finitely generated groups with Følner function growing faster than any given function, even non-subrecursive. In particular, the Følner sets of those groups are missing any algorithmic description. Analogous results were recovered in [15, 31].

But the behaviour for finitely presented groups remained open:

Question. [15, p.578, Gromov] “(d) Is there an universal bound on the asymptotic growth of the Følner functions of finitely presented amenable groups by a recursive (primitively recursive?) function? (Maybe there is such a bound in every given recursive class of presentations?). Or, at another extreme, are there finitely presented amenable groups with so fast growing Følner function, such that their amenability is unprovable in Arithmetic? (An enticing possibility would be this situation for the Thompson group).”

The above-mentioned possibility about Thompson group was studied in [28]: if the Thompson group F is amenable then its Følner function grows faster than any iterated exponential. For recursively presented groups, in [11] Erschler showed that the asymptotics of the Følner function of the k -iterated wreath-product of \mathbb{Z} is the k -th tetration of n .

One of our main results (Section 3) is the following partial answer to the aforementioned question of Gromov:

Theorem A. *The Følner function of a recursively presented amenable group is subrecursive. Moreover, every recursively enumerable class of recursive amenable presentations admits a uniform recursive upper bound for the asymptotic growth of the corresponding Følner functions.*

The main tool used in the proof of the above theorem is the construction of a uniform algorithm $\widehat{\mathcal{R}}$, described in Theorem 3.3, that for any $n \in \mathbb{N}$ and any recursive presentation, provides, if it exists, a function on the associated free group whose pushforward on the group is n -invariant (an equivalent notion for amenability, see Section 2). Let us fix some notation.

With any finite set X of generators of Γ , we associate a set X and a bijection $\varphi: X \rightarrow X$, $x \mapsto \varphi(x) := x$. We denote by \mathbb{F}_X the free group generated by X , and by $\pi_\Gamma: \mathbb{F}_X \rightarrow \Gamma$ the unique epimorphism extending φ . Γ has *solvable Word Problem* (WP) if there exists an algorithm that for every $\omega \in \mathbb{F}_X$ as an input, stops and establishes whether or not ω represents the identity in Γ (i.e.

$\pi_\Gamma(\omega) = 1_\Gamma$). This is equivalent to saying that $\ker \pi_\Gamma \subset \mathbb{F}_X$ is *recursive*. We also say that Γ is recursively (resp. finitely) presentable if there exists $R \subset \mathbb{F}_X$ recursive (resp. finite), such that the normal closure $R^{\mathbb{F}_X} = \ker \pi_\Gamma$. Dehn in [8] first formulated the Word Problem, several years before the study about computability started. Only in the fifties [2, 29] examples of finitely presented groups with unsolvable WP appeared.

From a practical point of view, often in computer science it is not important the behaviour of an algorithm for the totality of the inputs, because it is possible that it is strongly influenced by a small, negligible, subset of inputs. Sometimes it is more interesting to study the *average* or the behaviour for *most* of the inputs. This concept was developed even in group theory [1, 6, 16, 30]: in the Introduction of [19] it is well explained with the right references. In particular, Kapovich, Myasnikov, Schupp and Shpilrain formally defined the concept of *generic computability* and *generic-case complexity*, especially focusing on algorithmic problems for finitely generated groups. We now present the *generic Equality Problem*.

Following [25], we say that the Equality Problem (EP) is solvable on a subset $S \subset \mathbb{F}_X$ if there exists an algorithm with input $(\omega_1, \omega_2) \in \mathbb{F}_X \times \mathbb{F}_X$, such that whenever $(\omega_1, \omega_2) \in S \times S$ the algorithm stops, establishing whether $\pi_\Gamma(\omega_1) = \pi_\Gamma(\omega_2)$ or not. Notice that when S is a subgroup, EP is equivalent to the Word Problem for S .

Denoting by B_n the ball of radius n in \mathbb{F}_X , a subset $S \subset \mathbb{F}_X$ is called *generic* if

$$\lim_{n \rightarrow \infty} \frac{|S \cap B_n|}{|B_n|} = 1;$$

a subset is *negligible* if its complement is generic.

Definition. Γ has solvable generic EP if there exist a finite set of generators Y and a generic subset $S \subset \mathbb{F}_Y$ such that the EP is solvable on S .

The transition to genericity makes solvable some classical unsolvable problems; the literature in this direction is very rich, starting from [19] to [9, 18, 20, 21]. But, not less important, especially for cryptography, is to produce examples [12, 18, 25, 26] of problems generically hard or even generically undecidable. Up to now there were no examples of finitely presented groups with unsolvable generic WP or unsolvable generic EP. Here we provide examples of the latter by proving a sort of “stability” for the Word Problem in recursively presented amenable groups (this is the main result in Section 4):

Theorem B. *In the class of recursively presented amenable groups:*

$$\text{solvable WP} \iff \text{solvable generic EP}$$

To prove this, we use a variation of the algorithm $\widehat{\mathfrak{R}}$, and the following: computability, for every n , of a one-to-one preimage of an n -Følner set in a recursively presented group, gives solvability of WP (Theorem 3.1). Thus, more generally, solvability of EP on a set containing a preimage of a Følner sequence implies solvability of the WP.

As a byproduct, we answer Problem 1.5, b, in [25], since the finitely presented groups $G(M)$ in [17, 22] (Kharlampovich groups), are solvable and therefore amenable, with unsolvable Word Problem. This provides examples of finitely presented groups with unsolvable generic EP. Moreover, in [19] it is proved (linear) solvability of the generic Word Problem for solvable groups. Thus even if the Equality Problem is the natural generalization of the Word Problem, however the generic EP is different from the generic WP.

Let \mathcal{C}_A denote the class of recursively presented amenable groups and consider the following subclasses: \mathcal{C}_{WP} (with solvable WP), \mathcal{C}_{CF} (with computable Følner sets), \mathcal{C}_{CFI} (with computable Følner sets by one-to-one preimages), \mathcal{C}_{CR} (with computable Reiter functions), \mathcal{C}_{SF} (with subrecursive Følner function) (see next section for the definitions).

The following theorem summarizes the current understanding about the relations among these several notions of effective amenability.

Theorem C.

$$\mathcal{C}_{CFI} = \mathcal{C}_{WP} \subsetneq \mathcal{C}_{CF} \subset \mathcal{C}_{SF} = \mathcal{C}_A$$

The inclusions were already proved in [3, 4], the equalities are proved in Section 3.

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2. PRELIMINARIES

Throughout this paper Γ is a group generated by a finite set X , in bijection with X . The canonical epimorphism $\pi_\Gamma: \mathbb{F}_X \rightarrow \Gamma$ is the unique homomorphism such that $\pi_\Gamma(x) = x$ for every $x \in X$. Given an element ω in the free group \mathbb{F}_X we denote by $|\omega|$ the natural word length of ω with respect to $X \cup X^{-1}$; we denote by $B_n := \{\omega \in \mathbb{F}_X : |\omega| \leq n\}$ the ball of radius n and by $S_n := B_n \setminus B_{n-1} \subset \mathbb{F}_X$, the sphere of radius n . For a natural number k , we denote by

$[k] := \{1, 2, \dots, k\}$, and recall that $\mathfrak{Fol}_{\Gamma, X}(n)$ is the family of n -Følner sets of Γ with respect to X . The function χ_A is the characteristic function of the subset A (both for $A \subset \Gamma$ or $A \subset \mathbb{F}_X$).

Definition 2.1. A summable non-zero function $h: \Gamma \rightarrow \mathbb{R}^+$, $\|h\|_{1, \Gamma} := \sum_{g \in \Gamma} |h(g)| < \infty$, is n -invariant with respect to X if for all $x \in X$

$$(2) \quad \frac{\|h - {}_x h\|_{1, \Gamma}}{\|h\|_{1, \Gamma}} \leq n^{-1};$$

where ${}_x h: \Gamma \rightarrow \mathbb{R}^+$ is the function defined by ${}_x h(g) := h(x^{-1}g)$.

We denote by $\mathfrak{Reit}_{\Gamma, X}(n)$ (from the *Reiter condition* for amenability [32]) the set of all summable non-zero functions from Γ to \mathbb{R}^+ that are n -invariant with respect to X .

Remark 2.2. The following facts are well known and/or easy to prove (see [5, 7])

- $\Omega \in \mathfrak{Fol}_{\Gamma, X}(n) \implies \Omega g \in \mathfrak{Fol}_{\Gamma, X}(n), \forall g \in \Gamma;$
- $\Omega \in \mathfrak{Fol}_{\Gamma, X}(n) \implies \frac{|\Omega \setminus x^{-1}\Omega|}{|\Omega|} \leq \frac{1}{n}, \forall x \in X;$
- $\Omega \in \mathfrak{Fol}_{\Gamma, X}(n) \Leftrightarrow \frac{|\Omega \cap x\Omega|}{|\Omega|} \geq 1 - \frac{1}{n}, \forall x \in X;$
- $\Omega \in \mathfrak{Fol}_{\Gamma, X}(2n) \Leftrightarrow \chi_\Omega \in \mathfrak{Reit}_{\Gamma, X}(n),$
 since $\frac{\|\chi_\Omega - {}_x \chi_\Omega\|_{1, \Gamma}}{\|\chi_\Omega\|_{1, \Gamma}} = \frac{\|\chi_\Omega - \chi_{x\Omega}\|_{1, \Gamma}}{\|\chi_\Omega\|_{1, \Gamma}} = 2 \frac{|\Omega \setminus x\Omega|}{|\Omega|};$
- $h \in \mathfrak{Reit}_{\Gamma, X}(n) \implies \exists \Omega \subset \text{Supp}(h) := \{g \in \Gamma : h(g) \neq 0\}, \Omega \in \mathfrak{Fol}_{\Gamma, X}(n),$
 precisely, by the so-called layer cake decomposition, or Namioka's trick, there exists $\epsilon \in \mathbb{R}^+$
 such that $\{g \in \Gamma : h(g) > \epsilon\} \in \mathfrak{Fol}_{\Gamma, X}(n);$

Thus Γ is amenable if and only if $\mathfrak{Reit}_{\Gamma, X}(n) \neq \emptyset$ for every $n \in \mathbb{N}$ or, equivalently, $\mathfrak{Fol}_{\Gamma, X}(n) \neq \emptyset$ for every $n \in \mathbb{N}$. In order to define a notion of effective amenability for Γ we require the existence of an algorithm computing, in some sense, either Følner sets or Reiter functions. Since in general Γ has unsolvable Word Problem we “lift” the output to \mathbb{F}_X . The following notion was introduced and studied in [3, 4]:

Definition 2.3. Γ has *computable Følner sets* if there exists an algorithm with:

INPUT: $n \in \mathbb{N}$

OUTPUT: $F \subset \mathbb{F}_X$ finite, such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(n)$.

The computability of Følner sets does not depend on the choice of the finite set of generators and, in particular, for finitely presented groups, if we change a given finite presentation we can algorithmically update the algorithm.

The following is the analogue definition for the Reiter condition:

Definition 2.4. Γ has *computable Reiter functions* with respect to X if there exists an algorithm with

INPUT: $n \in \mathbb{N}$

OUTPUT: $f: \mathbb{F}_X \rightarrow \mathbb{Q}^+$, finitely supported, such that $\pi_{\Gamma*}(f) \in \mathfrak{Reit}_{\Gamma, X}(n)$,

where $\pi_{\Gamma*}(f): \Gamma \rightarrow \mathbb{Q}^+$ is the *pushforward* of f , defined by $\pi_{\Gamma*}(f)(g) := \sum_{\nu \in \pi_{\Gamma}^{-1}(g)} f(\nu)$.

Remark 2.5. Consider the commutative diagram of group epimorphisms:

$$\begin{array}{ccc} G_1 & \xrightarrow{\pi_1} & G_2 \\ & \searrow \pi_3 & \downarrow \pi_2 \\ & & G_3 \end{array}$$

and $f: G_1 \rightarrow \mathbb{R}$. Then the following holds:

- $\pi_{2*}(\pi_{1*}(f)) = \pi_{3*}(f)$ and if f is finitely supported then $\pi_{1*}(f): G_2 \rightarrow \mathbb{R}$ is finitely supported;
 - as a consequence, the definition of computability of Reiter functions does not depend on the choice of the finite set of generators;
 - $\pi_{1*}(gf) = \pi_{1(g)}\pi_{1*}(f)$, $\forall g \in G_1$;
 - $\|f\|_{1, G_1} \geq \|\pi_{1*}(f)\|_{1, G_2} \geq \|\pi_{3*}(f)\|_{1, G_3}$, and, if f is positive, equalities hold;
 - $\pi_{1*}(f) \in \mathfrak{Reit}_{G_2}(n) \implies \pi_{3*}(f) \in \mathfrak{Reit}_{G_3}(n)$,
- thus computability of Reiter functions passes to quotients.

3. EFFECTIVE AMENABILITY

Theorem 3.1. *The following are equivalent:*

- (i) Γ is amenable with solvable Word Problem;
- (ii) Γ is recursively presentable and there exists an algorithm with

INPUT: $n \in \mathbb{N}$

OUTPUT: $F \subset \mathbb{F}_X$ finite, such that $\pi_{\Gamma}(F) \in \mathfrak{Sol}_{\Gamma, X}(n)$ and $|F| = |\pi_{\Gamma}(F)|$.

Proof.

(i) \implies (ii)

Suppose Γ is amenable with solvable Word Problem. Then, by the latter property, for any given finite subset $F \subset \mathbb{F}_X$ we can algorithmically check if $\pi_{\Gamma}(F) \in \mathfrak{Sol}_{\Gamma, X}(n)$ and $|F| = |\pi_{\Gamma}(F)|$. Fixing an enumeration of the finite subsets of \mathbb{F}_X , we check these conditions until we find a correct F , whose existence is guaranteed by exists amenability of Γ .

Finally, solvability of the Word Problem ensures existence of a recursive set $R := \ker \pi_{\Gamma}$ of defining relations of Γ .

(ii) \implies (i)

It is clear that (ii) implies amenability of Γ . Let us show that Γ has solvable Word Problem.

For a given $\omega \in \mathbb{F}_X$, we denote by $n := \max\{|\omega|, 3\}$ and compute a finite subset F of \mathbb{F}_X such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(n^2)$ and $|F| = |\pi_\Gamma(F)| =: k$. We write $F =: \{f_1, f_2, \dots, f_k\}$ and $X =: \{x_1, x_2, \dots, x_d\}$. We consider $\Sigma_1^t, \dots, \Sigma_d^t \subset [k]^2$ for $t = 0, 1, \dots$ and set $\Sigma_\ell^0 = \emptyset$ for $\ell = 1, \dots, d$. The group Γ is recursively presented thus $\ker \pi_\Gamma$ is recursively enumerable, we list the trivial words η_1, η_2, \dots and update the Σ_ℓ^t 's in this way: we read $\eta_t \in \ker \pi_\Gamma$, for each $(i, j) \in [k]^2$ and each $\ell \in [d]$ such that $x_\ell f_i f_j^{-1} = \eta_t$ in \mathbb{F}_X , we set $\Sigma_\ell^t = \Sigma_\ell^{t-1} \cup \{(i, j)\}$; we stop when we meet \hat{t} such that $\min_\ell |\Sigma_\ell^{\hat{t}}| > (1 - \frac{1}{n^2})k$. We then simply write Σ_ℓ instead of $\Sigma_\ell^{\hat{t}}$.

Since $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(n^2)$, by Remark 2.2, we have that

$$(3) \quad \frac{|\{(i, j) : x_t f_i f_j^{-1} \in \ker \pi_\Gamma\}|}{k} \geq \frac{|\pi_\Gamma(F) \cap x_t \pi_\Gamma(F)|}{|\pi_\Gamma(F)|} > 1 - \frac{1}{n^2},$$

then this procedure will stop. The injectivity of π_Γ on F guarantees that if $(i, j), (i', j') \in \Sigma_\ell$ are distinct then $i \neq i'$ and $j \neq j'$. Then for $\ell = 1, \dots, d$ we can algorithmically choose $\sigma_\ell \in \text{Sym}(k)$, a permutation of $[k]$ such that $(i, j) \in \Sigma_\ell \implies \sigma_\ell(i) = j$.

Claim. The permutations $\sigma_1, \dots, \sigma_d$ have the following property

$$(4) \quad \ell_H(\omega(\sigma_1, \dots, \sigma_d)) \begin{cases} \leq \frac{1}{n}, & \text{if } \omega \in B_n \cap \ker \pi_\Gamma \\ \geq 1 - \frac{1}{n}, & \text{if } \omega \in B_n \setminus \ker \pi_\Gamma \end{cases}$$

where for $\sigma \in \text{Sym}(k)$ the positive real number $\ell_H(\sigma) := \frac{|\{i \in [k] : \sigma(i) \neq i\}|}{k}$ is the *normalized Hamming length* of σ .

Proof of the claim. Suppose $\omega = x_{l_n}^{s_n} \dots x_{l_2}^{s_2} x_{l_1}^{s_1}$, where $l_z \in [d]$, $s_z \in \{1, -1\}$ for $z = 1, 2, \dots, n$. We denote with $\Sigma_\ell^{-1} := \{(i, j) : (j, i) \in \Sigma_\ell\}$ and $N_\ell^{\pm 1} := \{i \in [k] : \nexists j \in [k] : (i, j) \in \Sigma_\ell^{\pm 1}\}$. We consider I_ω , the set of $i \in [k]$ for which we can compute $\omega(\sigma_1, \dots, \sigma_d)(i) = \sigma_{l_n}^{s_n} \dots \sigma_{l_2}^{s_2} \sigma_{l_1}^{s_1}(i)$ only looking at $\Sigma_1, \dots, \Sigma_d$:

$$I_\omega := \{i_0 \in [k] : \exists i_1, i_2, \dots, i_n \in [k] : (i_{t-1}, i_t) \in \Sigma_{l_z}^{s_z}, \forall z \in [n]\}$$

We can write $I_\omega = \{i_0 \in [k] : \sigma_{l_n}^{s_n} \dots \sigma_{l_2}^{s_2} \sigma_{l_1}^{s_1}(i_0) \notin N_{l_n}^{s_n}, \forall n' \in [n]\}$.

In order to estimate the cardinality of I_ω , we define $\phi: [k] \setminus I_\omega \hookrightarrow N_{l_n}^{s_n} \sqcup \dots \sqcup N_{l_2}^{s_2} \sqcup N_{l_1}^{s_1}$,

$\phi(i) := (n', i')$ where n' is the smallest number in $[n]$, such that $\sigma_{l_{n'}}^{s_{n'}} \dots \sigma_{l_2}^{s_2} \sigma_{l_1}^{s_1}(i) \in N_{l_{n'}}^{s_{n'}}$ and $i' = \sigma_{l_{n'}}^{s_{n'}} \dots \sigma_{l_2}^{s_2} \sigma_{l_1}^{s_1}(i)$. Since the map ϕ is injective, combining with the fact that, by construction of Σ_ℓ , $|N_\ell| \leq \frac{k}{n^2}$, we have

$$(5) \quad \begin{aligned} |[k] \setminus I_\omega| &\leq \sum_{z=1}^n |N_{l_z}^{s_z}| \leq \frac{k}{n}, \\ |I_\omega| &\geq (1 - \frac{1}{n})k. \end{aligned}$$

By construction of the Σ_ℓ 's, for $i \in I_\omega$, we have $\pi_\Gamma(f_{\omega(\sigma_1, \dots, \sigma_d)(i)}) = \pi_\Gamma(\omega(x_1, \dots, x_d)f_i)$. If $\omega \in \ker \pi_\Gamma$, for $i \in I_\omega$, we have $\pi_\Gamma(f_{\omega(\sigma_1, \dots, \sigma_d)(i)}) = \pi_\Gamma(f_i)$. By injectivity of π_Γ on F , i is a fixed point of $\omega(\sigma_1, \dots, \sigma_d)$; by virtue of estimation (5), we have $\ell_H(\omega(\sigma_1, \dots, \sigma_d)) \leq \frac{||k||I_\omega|}{k} \leq \frac{1}{n}$. If $\omega \notin \ker \pi_\Gamma$, for $i \in I_\omega$, we have $\pi_\Gamma(f_{\omega(\sigma_1, \dots, \sigma_d)(i)}) \neq \pi_\Gamma(f_i)$ and then I_ω contains only non-fixed points, and therefore, by virtue of estimation (5), $\ell_H(\omega(\sigma_1, \dots, \sigma_d)) \geq \frac{|I_\omega|}{k} \geq 1 - \frac{1}{n}$. This ends the proof of the claim. \square

We are now in position to complete the proof of the theorem. Since the number $\ell_H(\omega(\sigma_1, \dots, \sigma_d))$ is computable, by property (4) we can algorithmically determine whether ω belongs to $\ker \pi_\Gamma$ or not: thus Γ has solvable Word Problem (in the terminology of [3] we actually proved that Γ has *computable sofic approximations*, see Theorem 3.3.1 in [3].) \square

Proposition 3.2. *Suppose that Γ has solvable Word Problem. Then the following are equivalent:*

- (i) Γ is amenable;
- (ii) there exists an algorithm with
INPUT: $n \in \mathbb{N}$
OUTPUT: $F \subset \mathbb{F}_X$ finite, such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(n)$ and $|F| = |\pi_\Gamma(F)|$;
- (iii) Γ has computable Følner sets;
- (iv) Γ has computable Reiter functions;
- (v) Γ has subrecursive Følner function.

Proof. By virtue of Theorem 3.1 we have (i) \implies (ii). It is obvious that (ii) \implies (iii) \implies (i) and that (ii) \implies (v) \implies (i); by Remark 2.2 we have (iv) \implies (i). Finally (ii) \implies (iv) because if $F \subset \mathbb{F}_X$ is finite, such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(2n)$ and $|F| = |\pi_\Gamma(F)|$ then the pushforward of the characteristic function χ_F of F is the characteristic function $\chi_{\pi_\Gamma(F)}$ of $\pi_\Gamma(F)$: this is n -invariant by Remark 2.2. \square

Theorem 3.3. *Suppose that Γ is recursively presentable. Then the following are equivalent:*

- (i) Γ is amenable;
- (ii) Γ has subrecursive Følner function;
- (iii) there exists an algorithm with
INPUT: $n \in \mathbb{N}$
OUTPUT: $F \subset \mathbb{F}_X$ finite, such that $\pi_\Gamma(F)$ contains an n -Følner set;
- (iv) Γ has computable Reiter functions.

Proof. It is clear that (iii) \implies (ii) \implies (i);
 (iv) \implies (iii)

For every $n \in \mathbb{N}$ the output of the algorithm in Definition 2.4 is a function $f: \mathbb{F}_X \rightarrow \mathbb{Q}^+$ with finite support, say $F \subset \mathbb{F}_X$. Let $h := \pi_{\Gamma*}(f)$ be the pushforward of f , so that $h \in \mathfrak{Reit}_{\Gamma, X}(n)$. Then, as mentioned in Remark 2.2, there exists $\epsilon \in \mathbb{R}^+$ such that $\Omega_\epsilon := \{g \in \Gamma : h(g) > \epsilon\} \in \mathfrak{Fol}_{\Gamma, X}(n)$. We complete by observing that $\Omega_\epsilon \subset \pi_\Gamma(F)$.

(i) \implies (iv)

The first step is to write, fixing $n \in \mathbb{N}$, a subroutine $\mathfrak{K}(n)$ that, taken a function $f: \mathbb{F}_X \rightarrow \mathbb{Q}^+$ with finite support $F \subset \mathbb{F}_X$, stops if $\pi_{\Gamma*}(f) \in \mathfrak{Reit}_{\Gamma, X}(n)$. In fact, even if we cannot compute the pushforward (because we have no assumptions on WP), we can estimate the n -invariance after the following arguments.

With every partition \mathcal{Q} of the finite support F we associate the positive rational numbers

$$M_{\mathcal{Q}}^x(f) := \frac{\sum_{V \in \mathcal{Q}} |\sum_{\nu \in V} (f(\nu) - f(x^{-1}\nu))|}{\sum_{\nu \in F} f(\nu)}, \quad x \in X.$$

Denoting by \mathcal{P} the canonical partition of F associated with π_Γ ($\forall \nu_1, \nu_2 \in F$ there exist $V \in \mathcal{P}$ such that $\nu_1, \nu_2 \in V$ if and only if $\pi_\Gamma(\nu_1) = \pi_\Gamma(\nu_2)$), we have

$$(6) \quad \frac{\|\pi_{\Gamma*}(f) - x\pi_{\Gamma*}(f)\|_{1, \Gamma}}{\|\pi_{\Gamma*}(f)\|_{1, \Gamma}} = M_{\mathcal{P}}^x(f), \quad \forall x \in X.$$

By the triangle inequality, for any two partitions \mathcal{Q} and \mathcal{Q}' of F if $\mathcal{Q} \leq \mathcal{Q}'$ then $M_{\mathcal{Q}}^x(f) \geq M_{\mathcal{Q}'}^x(f)$.

In particular for any partition \mathcal{P}' of F such that $\mathcal{P}' \leq \mathcal{P}$, or equivalently, such that $\nu_1, \nu_2 \in V \in \mathcal{P}' \implies \pi_\Gamma(\nu_1) = \pi_\Gamma(\nu_2)$, using equation (6) we have

$$(7) \quad \frac{\|\pi_{\Gamma*}(f) - x\pi_{\Gamma*}(f)\|_{1, \Gamma}}{\|\pi_{\Gamma*}(f)\|_{1, \Gamma}} \leq M_{\mathcal{P}'}^x(f), \quad \forall x \in X.$$

So we define $\mathfrak{K}(n)$ as follows: with input f , it sets $\mathcal{P}_0 := \{\{f\} : f \in F\}$, the finest partition of F . As Γ is recursively presented, there is a recursive enumeration η_1, η_2, \dots of the words in $\ker \pi_\Gamma$. When $\mathfrak{K}(n)$ reads η_m , for every pair of distinct $V_1, V_2 \in \mathcal{P}_{m-1}$ such that $\eta_m \in V_1 V_2^{-1}$, it merges V_1 and V_2 , defining a new partition \mathcal{P}_m ; then it computes $M_{\mathcal{P}_m}^x(f)$ and, if $M_{\mathcal{P}_m}^x(f) \leq n^{-1}$ for every $x \in X$, it stops, if not, it goes to the next trivial word η_{m+1} .

By construction $\mathcal{P}_m \leq \mathcal{P}$ and the inequality (7) holds (with $\mathcal{P}' = \mathcal{P}_m$); thus, when $\mathfrak{K}(n)$ stops, $M_{\mathcal{P}_m}^x(f) \leq n^{-1}$ for every $x \in X$, and therefore $\pi_{\Gamma*}(f)$ is n -invariant. Conversely, if $\pi_{\Gamma*}(f)$ is n -invariant, at latest when $\mathcal{P}_m = \mathcal{P}$ we have $M_{\mathcal{P}_m}^x(f) \leq n^{-1}$, for any $x \in X$, by equality (6).

Now, using hypothesis (i), for every $n \in \mathbb{N}$ there exists a non-empty finite subset $F \in \mathbb{F}_X$ such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, X}(2n)$ and $|F| = |\pi_\Gamma(F)|$: the pushforward of the characteristic function χ_F of F is the characteristic function $\chi_{\pi_\Gamma(F)} \in \mathfrak{Reit}_{\Gamma, X}(n)$, by Remark 2.2. We list all finite subsets of \mathbb{F}_X : F_1, F_2, \dots (they are countably many) and we simultaneously run $\mathfrak{K}(n)$ on $\chi_{F_1}, \chi_{F_2} \dots$ until

one of the subroutines stops, providing a function with n -invariant pushforward (the seeked Reiter funtion). \square

In combination with Theorem 3.1 and the results in [4], this proves the Theorem C in the Introduction.

Remark 3.4. In general, the algorithm $\mathfrak{R}(n)$ may stop also with a function χ_F whose pushforward is not a characteristic function in Γ : this obstruction to reach n -Følner sets cannot be avoided because if we could change $\mathfrak{R}(n)$ in order to stop only when $\pi_{\Gamma*}(\chi_F)$ is characteristic, by Theorem 3.1 this would imply that Γ has solvable Word Problem, and this is not true even for finitely presented groups with subrecursive Følner function.

The question -whether we can obtain computability of Følner sets (i.e. of a preimage not necessarily 1-1) with a similar algorithm- remains open: actually, we can estimate better and better $|\pi_{\Gamma}(F) \setminus x\pi_{\Gamma}(F)|$ from above listing the elements in $\ker \pi_{\Gamma}$, but in this case the denominator $|\pi_{\Gamma}(F)|$ is not computable and, at least for a general set, it is impossible to estimate from below its cardinality without solvability of the Word Problem. The same issue appears for stability of computability of Følner sets under quotients, see [4].

Consider an enumeration $(P_i)_{i \in \mathbb{N}}$ of all finitely generated recursive presentations, $P_i = \{X_i | R_i\}$, $\Gamma_i := \mathbb{F}_{X_i} / R_i^{\mathbb{F}_{X_i}}$. Clearly, we can extend \mathfrak{R} to the universal algorithm $\widehat{\mathfrak{R}}$, that taking as an input n and a presentation P_i , runs as $\mathfrak{R}(n)$ on \mathbb{F}_{X_i} , using only the recursive set of relations R_i , and stops if the group Γ_i admits n -Følner sets with respect to X_i .

Corollary 3.5. *There exists a 2-place partial recursive function \mathcal{U} such that*

$$F_{\Gamma_i, X_i}(n) \leq \mathcal{U}(i, n)$$

on the domain $\{(i, n) \in \mathbb{N}^2 : F_{\Gamma_i, X_i}(n) < \infty\}$.

Corollary 3.6. *For every $n \in \mathbb{N}$ fixed, the set of finitely generated recursive presentations of groups admitting n -Følner sets is recursively enumerable.*

Remark 3.7. For every $n \in \mathbb{N}$ fixed the property of admitting n -Følner sets is a presentation property, not a group property.

Corollary 3.8. *For every recursively enumerable class \mathcal{C} of finitely generated recursive presentations of amenable groups there exists a recursive function $U_{\mathcal{C}}$ such that for every $P_i \in \mathcal{C}$:*

$$F_{\Gamma_i, X_i} \leq U_{\mathcal{C}} \text{ eventually.}$$

Proof. More generally, suppose that $(f_i)_{i \in \mathbb{N}}$ is a recursively enumerable set of recursive functions $f_i: \mathbb{N} \rightarrow \mathbb{N}$. Then the function $U: \mathbb{N} \rightarrow \mathbb{N}$, defined as

$$U(n) := \max_{i \leq n} f_i(n)$$

is recursive and eventually dominates f_i , for every $i \in \mathbb{N}$. □

This concludes the proof of Theorem A in the Introduction.

4. GENERIC EP

Lemma 4.1. *Suppose Γ has solvable Equality Problem on S , where $S \subset \mathbb{F}_X$. Then there exists a family \mathcal{A} of finite subsets of \mathbb{F}_X , with the following properties:*

- (1- \mathcal{A}) \mathcal{A} is recursively enumerable;
- (2- \mathcal{A}) $\pi_{\Gamma|_A}$ is injective $\forall A \in \mathcal{A}$;
- (3- \mathcal{A}) $\forall S' \subset S$, S' finite, $\exists A \in \mathcal{A}$ such that $\pi_{\Gamma}(A) = \pi_{\Gamma}(S')$.

Proof. Let \mathfrak{A} be the associated algorithm for the solvability of the Equality Problem. Recall that \mathfrak{A} (at least) stops on $S \times S$. We can easily define an algorithm \mathfrak{A}' with input B , any finite subset of \mathbb{F}_X , that checks if any two words in B represent the same elements in Γ , that is, it checks if $\pi_{\Gamma|_B}$ is injective. Clearly \mathfrak{A}' stops at least for every finite $B \subset S$. Thus we enumerate all finite subsets of \mathbb{F}_X : B_1, B_2, \dots and we simultaneously (diagonally) run \mathfrak{A}' on these sets, and give as an output only those subsets B for which the two following conditions are met: \mathfrak{A}' stops and \mathfrak{A}' has checked that $\pi_{\Gamma|_B}$ is injective. \mathcal{A} is the set of these outputs. Properties (1- \mathcal{A}) and (2- \mathcal{A}) hold by construction of \mathcal{A} . For any finite $S' \subset S$, for each element of $\pi_{\Gamma}(S')$ we choose only one representative word in S' , obtaining a subset $A \subset S' \subset S$ such that $\pi_{\Gamma}(A) = \pi_{\Gamma}(S')$ and $\pi_{\Gamma|_A}$ is injective. Then $A \in \mathcal{A}$ and the property (3- \mathcal{A}) is proved. □

Lemma 4.2. *[Upper Banach genericity] Suppose that S is a generic subset of \mathbb{F}_X . Then for every finite subset $F \subset \mathbb{F}_X$ there exist $y \in \mathbb{F}_X$ such that $Fy \subset S$.*

Proof. Since for every finite set F there exist $k \in \mathbb{N}$ such that $F \subset B_k$, without loss of generality we may reduce to the case $F = B_k$. We denote by $N := S^c$, the complement of S ; so that, being S generic, N is *negligible*, that is $\frac{|N \cap B_n|}{|B_n|} \rightarrow 0$. We want to prove that there exists $y \in \mathbb{F}_X$ such that $N \cap B_k y = \emptyset$. Recall that $S_n := B_n \setminus B_{n-1}$ is the n -sphere in \mathbb{F}_X .

For every $m \in \mathbb{N}$ we have:

$$B_{m+2k} \supset \bigsqcup_{\omega \in S_m} B_k a_{\omega} \omega,$$

where, for every $\omega \in S_m$, the word a_ω is a suitable element of S_k such that $|a_\omega\omega| = m + k$. Let's check the disjointness of the union. For all distinct $\omega, \omega' \in S_m$, since $|\omega\omega'^{-1}| \geq 2$ we have $|a_\omega\omega\omega'^{-1}a_{\omega'}^{-1}| \geq 2k + 2$. By the triangular inequality, $B_ka_\omega\omega$ and $B_ka_{\omega'}\omega'$ are disjoint.

Suppose, by contradiction, that $N \cap B_ky \neq \emptyset$ for every y , then we have

$$(8) \quad \frac{|B_n \cap N|}{|B_n|} \geq \frac{|S_{n-2k}|}{|B_n|} \rightarrow \frac{2|X| - 2}{(2|X| - 1)^{2k+1}}.$$

If $|X| \geq 2$, this is impossible since the set N is negligible.

If $|X| = 1$, we notice that in B_n there are approximately $\frac{n}{k}$ disjoint copies of B_k and the last term in (8) equals $\frac{1}{k}$, providing again a contradiction. \square

Remark 4.3. Upper Banach genericity of Lemma 4.2 also holds if we replace a generic set S by a set $T \subset \mathbb{F}_X$ such that $\frac{|T \cap S_n|}{|S_n|} \rightarrow 1$. Moreover, upper Banach genericity is strictly weaker than genericity: fixing $x \in X$, for any function $f: \mathbb{N} \rightarrow \mathbb{N}$ the subset $T_f := \bigcup_{n \in \mathbb{N}} B_n x^{f(n)}$ clearly contains an increasing sequence of translated balls but the asymptotic behaviour of the ratio $\frac{|T_f \cap B_n|}{|B_n|}$ can be arbitrary (depends on the choice of f).

Lemma 4.4. *Suppose that Γ is amenable and $S \subset \mathbb{F}_X$ is generic. Then $\pi_\Gamma(S)$ contains a Følner sequence: $\forall n \in \mathbb{N} \exists \Omega_n \in \mathfrak{Fol}_{\Gamma, X}(n)$ such that $\Omega_n \subset \pi_\Gamma(S)$.*

Proof. Since Γ is amenable, for every $n \in \mathbb{N}$ there exists a finite subset $F_n \subset \mathbb{F}_X$ such that $\pi_\Gamma(F_n) \in \mathfrak{Fol}_{\Gamma, X}(n)$. Since S is generic, then by virtue of Lemma 4.2 there exists $y_n \in \mathbb{F}_X$ such that $F_n y_n \subset S$; by Remark 2.2, the set $\Omega_n := \pi_\Gamma(F_n y_n) \in \mathfrak{Fol}_{\Gamma, X}(n)$. \square

We conclude with the following theorem (cf. Theorem B in the Introduction).

Theorem 4.5. *Suppose that Γ is amenable and recursively presentable. Then the following are equivalent:*

- (i) Γ has solvable Word Problem;
- (ii) Γ has solvable generic Equality Problem.

Proof. (i) \implies (ii) is true in general.

(ii) \implies (i)

By virtue of Theorem 3.1, it is enough to show the existence of a finite generating set Y and an algorithm with:

INPUT: $n \in \mathbb{N}$

OUTPUT: $F \subset \mathbb{F}_Y$ finite, such that $\pi_\Gamma(F) \in \mathfrak{Fol}_{\Gamma, Y}(n)$ and $|F| = |\pi_\Gamma(F)|$.

Since Γ has solvable generic Equality Problem, there exists a set of generators, say Y , and a generic subset $S \subset \mathbb{F}_Y$ with solvable EP.

Let \mathcal{A} be the family given by Lemma 4.1. By property (1- \mathcal{A}) we have a recursive enumeration of \mathcal{A} : E_1, E_2, \dots . Thanks to property (3- \mathcal{A}), the family $\pi_\Gamma(\mathcal{A}) := \{\pi_\Gamma(E_1), \pi_\Gamma(E_2), \dots\}$ contains $\{\pi_\Gamma(S') : S' \subset S, S' \text{ finite}\}$ and, by Lemma 4.4, for every $n \in \mathbb{N}$ we have

$$\pi_\Gamma(\mathcal{A}) \cap \mathfrak{Fol}_{\Gamma,Y}(n) \neq \emptyset.$$

The property (2- \mathcal{A}) ensures that $\pi_{\Gamma*}(\chi_{E_i}) = \chi_{\pi_\Gamma(E_i)}$, and therefore by Remark 2.2, for all $n \in \mathbb{N}$

$$\{\pi_{\Gamma*}(\chi_{E_1}), \pi_{\Gamma*}(\chi_{E_2}), \dots\} \cap \mathfrak{Reit}_{\Gamma,Y}(n) \neq \emptyset.$$

We now are in position to define the seeked algorithm:

for every $n \in \mathbb{N}$ we run the algorithm $\mathfrak{R}(n)$ used in the proof of Theorem 3.3, simultaneously on the functions $\chi_{E_1}, \chi_{E_2}, \dots$ until one of the subroutines stops, providing a function χ such that $\pi_{\Gamma*}(\chi) \in \mathfrak{Reit}_{\Gamma,Y}(n)$. Again, by the property (2- \mathcal{A}), the pushforward $\pi_{\Gamma*}(\chi)$ is still a characteristic function and then by Remark 2.2, the output $F := \text{Supp}(\chi)$ (i.e. $\chi = \chi_F$) satisfies the required conditions. \square

Corollary 4.6. *The finitely presented groups $G(M)$ of [17, 22], have unsolvable generic Equality Problem.*

5. QUESTIONS AND FINAL REMARKS

The existence of a recursive universal bound for recursively (resp. finitely) presented amenable groups can be related to the arithmetic hierarchy of the property of being amenable. But there's no hope to establish, using our algorithm, if the bound is primitively recursive, since the stopping time depends on the bound itself.

Question. Is the class of recursively (finitely) presented amenable groups recursively enumerable?

For solvable groups the question is open (see [27]), even if in this case a universal bound for Følner functions of groups of this class is known [33]. In [14] there are some questions and remarks about decidability of amenability and bounds for Følner function in some subclasses of groups.

The Kharlampovich groups $G(M)$ have:

- unsolvable Word Problem [17];
- solvable generic Word Problem [19];
- unsolvable strongly generic Word Problem [12];
- unsolvable generic Equality Problem (Corollary 4.6);
- computable Følner sets [3, 4].

Here a subset $S \subset \mathbb{F}_X$ is *strongly generic* if $\frac{|S \cap B_n|}{|B_n|} \rightarrow 1$ exponentially fast, and a strongly generic problem is solvable if it is solvable on a strongly generic set (for some generating set).

Question. Does solvability of the strongly generic WP imply solvability of the (strongly) generic EP?

If we measure genericity for the Equality Problem in $\mathbb{F}_X \times \mathbb{F}_X$, with a general subset, not necessarily of type $S \times S$, it is possible to prove:

Claim. If Γ has solvable strongly generic WP then there exists $T \subset \mathbb{F}_X \times \mathbb{F}_X$, $\frac{|T \cap (B_n \times B_n)|}{|B_n \times B_n|} \rightarrow 1$ exponentially fast, such that EP is solvable on T .

This is an immediate consequence of Lemma 3.2 in [12]. But, with this weaker notion of genericity for the EP it is not clear if we can reach the analogous thesis of Theorem 4.5 (this way, we would recover Theorem 2.3 in [12]).

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